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HW 7

Solutions

① (a)

Suppose $c_1\vec{a}=\vec{0}$.

Then $c_1\langle -1,-1\rangle=\langle 0,0\rangle$.

Then $\langle -c_1,-c_1\rangle=\langle 0,0\rangle$

So, $c_1=0$.

Thus, $\{\vec{a}\}$ is a lin. ind. set.

So we can make $W=\text{span}(\vec{a})$

and $\beta=[\vec{a}]$ is a basis for W.

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① (b)

$W=\text{span}(\vec{a})=\{c_1\vec{a} \mid c_1\in\mathbb{R}\}$.

Some vectors in W are:

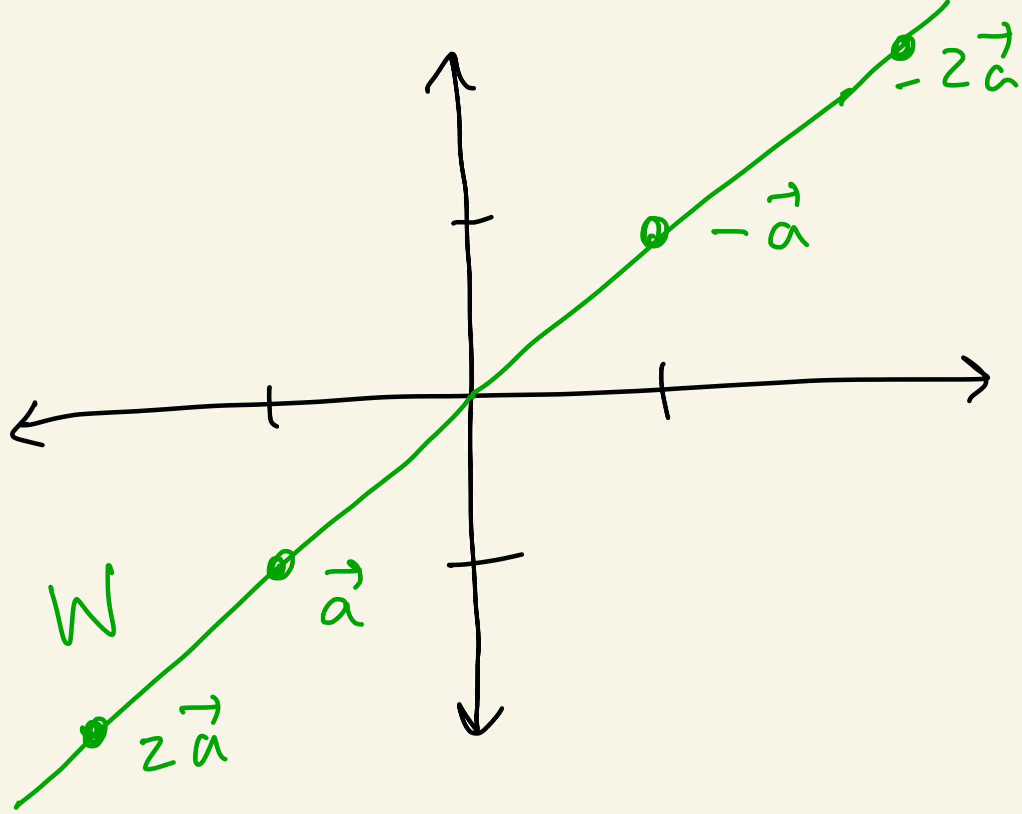
$2\cdot\vec{a}=2\langle -1,-1\rangle=\langle -2,-2\rangle$

$-\vec{a}=-\langle -1,-1\rangle=\langle 1,1\rangle$

$\frac{1}{2}\vec{a}=\frac{1}{2}\langle -1,-1\rangle=\langle -\frac{1}{2},-\frac{1}{2}\rangle$

$0\vec{a}=0\langle -1,-1\rangle=\langle 0,0\rangle$

1 (c)



1 (d)

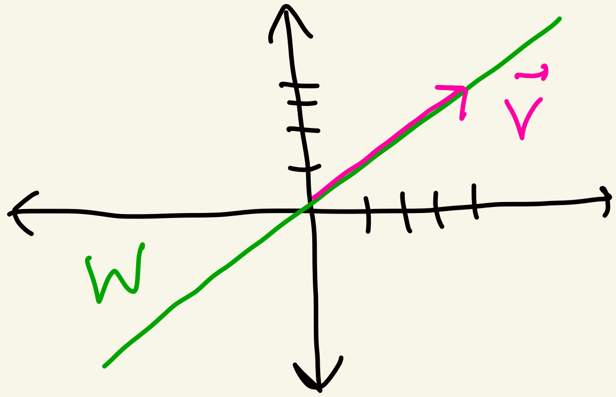
$\dim(W) = 1$

since β has 1 vector in it

1 (e)

$\vec{v} = \langle 4, 4 \rangle = -4 \langle -1, -1 \rangle = -4\vec{a}$

Since $\vec{v} = -4\vec{a}$ we know that $\vec{v}$ is in $W = \text{span}(\vec{a})$.



$\boxed{①(f)}$ Can we solve

$\vec{v} = c_1 \vec{a}$?

We would need

$\langle 1, \frac{1}{2} \rangle = c_1 \langle -1, -1 \rangle$

which would require

$\langle 1, \frac{1}{2} \rangle = \langle -c_1, -c_1 \rangle$

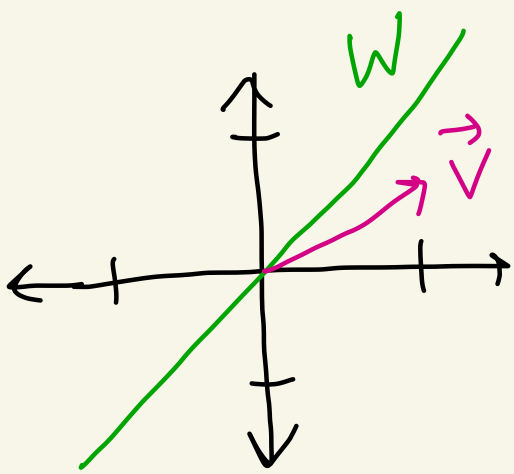
So, $-c_1 = 1$, $-c_1 = \frac{1}{2}$.

Then $c_1 = -1$ and $c_1 = \frac{-1}{2}$

This is impossible.

Thus, $\vec{v}$ is not in

$W = \text{span}(\vec{a})$.



$\boxed{②\ (a)}$

Suppose $c_1\vec{i} + c_2\vec{k} = \vec{0}$.

Then, $c_1\langle 1,0,0\rangle + c_2\langle 0,0,1\rangle = \langle 0,0,0\rangle$

Then, $\langle c_1,0,0\rangle + \langle 0,0,c_2\rangle = \langle 0,0,0\rangle$

So, $\langle c_1,0,c_2\rangle = \langle 0,0,0\rangle$.

Thus, $c_1 = 0$, $c_2 = 0$.

Since the only solution to $c_1\vec{i} + c_2\vec{k} = \vec{0}$ is $c_1 = 0, c_2 = 0$ we know that $\vec{i}, \vec{k}$ are linearly independent vectors.

Thus, $B = [\vec{i}, \vec{k}]$ is a basis for $W = \text{span}(\vec{i}, \vec{k})$.

$\boxed{②\ (b)}$

$W = \text{span}(\vec{i}, \vec{k})$

$= \{ c_1 \vec{i} + c_2 \vec{k} \mid c_1, c_2 \in \mathbb{R} \}$

Some vectors in W are

$2\vec{i} - 3\vec{k} = 2\langle 1,0,0\rangle - 3\langle 0,0,1\rangle = \langle 2,0,-3\rangle$

$\vec{i} + 0\vec{k} = \langle 1,0,0\rangle + 0\langle 0,0,1\rangle = \langle 1,0,0\rangle$

$-\vec{i} - 10\vec{k} = -\langle 1,0,0\rangle - 10\langle 0,0,1\rangle = \langle -1,0,-10\rangle$

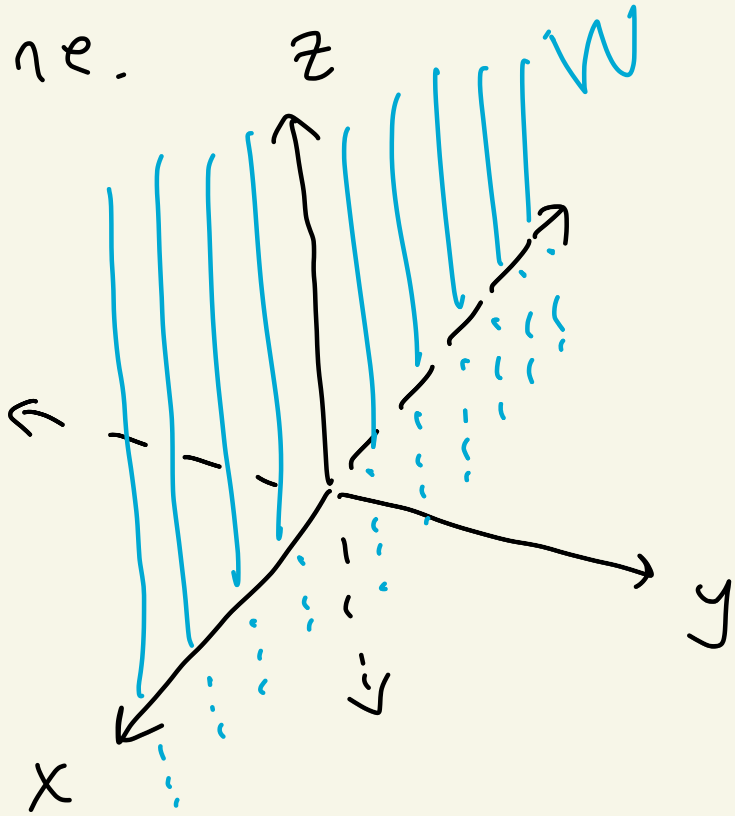
$5\vec{i} + 2\vec{k} = 5\langle 1,0,0\rangle + 2\langle 0,0,1\rangle = \langle 5,0,2\rangle$

$\boxed{2(c)}$ The vectors in W are the ones of the form

$c_1\vec{i} + c_2\vec{k} = c_1\langle 1,0,0\rangle + c_2\langle 0,0,1\rangle$

$= \langle c_1, 0, c_2\rangle$

This is the plane $y=0$, ie these vectors lie on the xz-plane.



② (d) Since the basis $\beta = [\vec{i}, \vec{k}]$ for W has 2 vectors in it,

$\dim(W) = 2$.

2 (e)

$\vec{v} = \langle 3,0,2\rangle = \langle 3,0,0\rangle + \langle 0,0,2\rangle$

$= 3\langle 1,0,0\rangle + 2\langle 0,0,1\rangle$

$= 3\vec{i} + 2\vec{k}.$

Thus, $\vec{v}$ is in $W = \text{span}(\vec{i}, \vec{k}).$

2 (f) Suppose we tried to solve

$\vec{v} = c_1\vec{i} + c_2\vec{k}.$

Then we would need

$\langle 1,3,4\rangle = c_1\langle 1,0,0\rangle + c_2\langle 0,0,1\rangle$

which would require

$\langle 1,3,4\rangle = \langle c_1,0,0\rangle + \langle 0,0,c_2\rangle$

or

$\langle 1,3,4\rangle = \langle c_1,0,c_2\rangle$

But then $3=0$.

This is impossible.

So, $\vec{v}$ is not in $W=\text{span}(\vec{i},\vec{k})$.

$\boxed{3\ (a)}$

Suppose $c_1\vec{a}+c_2\vec{b}=\vec{0}$.

Then, $c_1\langle 1,1,1\rangle + c_2\langle 1,0,0\rangle = \langle 0,0,0\rangle$.

So, $\langle c_1,c_1,c_1\rangle + \langle c_2,0,0\rangle = \langle 0,0,0\rangle$.

Thus, $\langle c_1+c_2, c_1, c_1\rangle = \langle 0,0,0\rangle$.

This gives $c_1+c_2=0$, $c_1=0$, $c_1=0$.

Then, $c_1=0$, $c_2=-c_1=-0=0$.

Since the only solutions to

$c_1\vec{a}+c_2\vec{b}=\vec{0}$

are $c_1=0$, $c_2=0$ we know that

$\vec{a}$ and $\vec{b}$ are linearly independent.
Thus, $\beta = [\vec{a}, \vec{b}]$ is a basis for $W = \text{span}(\vec{a}, \vec{b})$.

$\boxed{\text{3(b)}}$

$$W = \text{span}(\vec{a}, \vec{b})$$
$$= \{ c_1\vec{a} + c_2\vec{b} \mid c_1, c_2 \in \mathbb{R} \}$$

Thus, some vectors in W are:

$$0\vec{a} + \vec{b} = 0\langle 1,1,1\rangle + \langle 1,0,0\rangle = \langle 1,0,0\rangle$$
$$-\vec{a} + 2\vec{b} = -\langle 1,1,1\rangle + 2\langle 1,0,0\rangle = \langle 1,-1,-1\rangle$$
$$2\vec{a} + 0\vec{b} = 2\langle 1,1,1\rangle + 0\langle 1,0,0\rangle = \langle 2,2,2\rangle$$
$$3\vec{a} + 5\vec{b} = 3\langle 1,1,1\rangle + 5\langle 1,0,0\rangle = \langle 8,3,3\rangle$$

$\boxed{\text{3(c)}}$ Since the basis $\beta = [\vec{a}, \vec{b}]$ for W has 2 vectors in it, $\dim(W) = 2$.

$\boxed{3(d)}$ We want to solve $\vec{v} = c_1\vec{a} + c_2\vec{b}$.

This requires $\langle \frac{1}{2}, -3, -3\rangle = c_1\langle 1,1,1\rangle + c_2\langle 1,0,0\rangle$.

This needs $\langle \frac{1}{2}, -3, -3\rangle = \langle c_1 + c_2, c_1, c_1\rangle$.

We get $c_1 + c_2 = \frac{1}{2}$, $c_1 = -3$, $c_1 = -3$.

So, $c_1 = -3$, $c_2 = \frac{1}{2} - c_1 = \frac{1}{2} - (-3) = \frac{7}{2}$.

Thus, $\vec{v} = -3\vec{a} + \frac{7}{2}\vec{b}$.

So, $\vec{v}$ is in $W = \text{span}(\vec{a}, \vec{b})$.

$\boxed{3(e)}$ We want to try to solve $\vec{v} = c_1\vec{a} + c_2\vec{b}$.

This becomes $\langle 1,2,3\rangle = c_1\langle 1,1,1\rangle + c_2\langle 1,0,0\rangle$.

This gives $\langle 1,2,3\rangle = \langle c_1 + c_2, c_1, c_1\rangle$

This gives $1 = c_1 + c_2$, $2 = c_1$, $3 = c_1$.

But $c_1 = 2$ and $c_1 = 3$ is impossible.

Thus, $\vec{v}$ is not in $W = \text{span}(\vec{a}, \vec{b})$.

$\boxed{4(a)}$ $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, 2x - y = 0 \right\}$

$\boxed{(i-iii)}$ By the homogeneous subspace theorem, W is a subspace of $\mathbb{R}^2$. Let's find a basis for W.

Let $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ be in W.

Then

$$\boxed{2x - y = 0}$$

Or

$$\boxed{x - \frac{1}{2}y = 0}$$

The solutions to this system are

$$y = t$$
$$x = \frac{1}{2}y = \frac{1}{2}t$$

Thus,

$$\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

So any vector $\vec{w}$ in W lies in the span of $\vec{a} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.

Since $\vec{a} \neq \vec{0}$, the set $\{\vec{a}\}$ is a linearly independent set.

Thus, $W = \text{span}(\vec{a})$ with basis $\beta = [\vec{a}]$

And $\dim(W) = 1$ since β consists of 1 vector.

$\boxed{\text{(iv)}}$ $W = \text{span}(\vec{a}) = \{c_1\vec{a} \mid c_1 \in \mathbb{R}\}$.

Here are 4 example vectors in W:

$$2\vec{a} = 2\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$0\vec{a} = 0\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3\vec{a} = -3\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -3 \end{pmatrix}$$

$$\pi\vec{a} = \pi\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \pi \end{pmatrix}$$

$\boxed{4(b)}$

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{r} x - y + 2z = 0 \\ y + z = 0 \end{array} \right\}$$

$\boxed{(i\text{-}iii)}$ By the homogeneous subspace theorem, W is a subspace of $\mathbb{R}^3$.

Let's find a basis for W.

Let $\vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in W.

Then,

$$\boxed{\begin{array}{r} x - y + 2z = 0 \\ y + z = 0 \end{array}} \begin{array}{l} ① \\ ② \end{array}$$

leading: x, y

free: z

Then,

$$z = t$$
$$y = -z = -t$$
$$x = y - 2z = -t - 2t = -3t$$

So,

$$\vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

So any vector $\vec{w}$ in W lies in the span of $\vec{a} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$.

Since $\vec{a}$ is a single non-zero vector, $\{\vec{a}\}$ is a linearly independent set.

Thus, $W = \text{span}(\vec{a})$ with basis $\beta = [\vec{a}]$.

So, $\dim(W) = 1$ since β has 1 vector in it.

$\boxed{(iv)}$ $W = \text{span}(\vec{a})$ where $\vec{a} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$.

Thus, four example vectors in W are:

$$2\vec{a} = 2\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 2 \end{pmatrix}$$

$$-\vec{a} = -\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\frac{1}{2}\vec{a} = \frac{1}{2}\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$0\vec{a} = 0\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

4(c) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, 2x - 4y - 3z = 0 \right\}$

(i-iii) By the homogeneous subspace theorem, W is a subspace of $\mathbb{R}^3$.

Let's find a basis for W.

Let $\vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in W.

Then,

$$\boxed{2x - 4y - 3z = 0}$$

So,

$$\boxed{x - 2y - \frac{3}{2}z = 0}$$

leading: x
free: y, z

Thus,

$$x = 2y + \frac{3}{2}z \quad ①$$
$$y = t \quad ②$$
$$z = u \quad ③$$

So,

③ $z = u$

② $y = t$

① $x = 2y + \frac{3}{2}z = 2t + \frac{3}{2}u$

Thus,

$$\vec{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t + \frac{3}{2}u \\ t \\ u \end{pmatrix}$$

$$= \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{2}u \\ 0 \\ u \end{pmatrix}$$

$$= t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

Thus, if $\vec{w}$ is in W, the $\vec{w}$ lies in the span of $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$.

Let's show that $\vec{a}$ & $\vec{b}$ are linearly independent.

Suppose $c_1\vec{a} + c_2\vec{b} = \vec{0}$.

Then, $c_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

So, $\begin{pmatrix} 2c_1 + \frac{3}{2}c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Thus, $2c_1 + \frac{3}{2}c_2 = 0$, $c_1 = 0$, $c_2 = 0$.

So, the only solutions to $c_1\vec{a} + c_2\vec{b} = \vec{0}$ are $c_1 = 0, c_2 = 0$.

Thus, $\vec{a}, \vec{b}$ are linearly independent

Therefore, $W = \text{span}(\vec{a}, \vec{b})$ where $\beta = [\vec{a}, \vec{b}]$ is a basis for W.

And $\dim(W) = 2$ since β has 2 vectors in it.

(iv) $W = \text{span}(\vec{a}, \vec{b}) = \{ c_1 \vec{a} + c_2 \vec{b} \mid c_1, c_2 \in \mathbb{R} \}$.

So, 4 example vectors in W are:

$$1 \cdot \vec{a} + 1 \cdot \vec{b} = 1 \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 \\ 1 \\ 1 \end{pmatrix}$$

$$0\vec{a} + 2\vec{b} = 0 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$10\vec{a} + 0\vec{b} = 10 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 0 \end{pmatrix}$$

$$1 \cdot \vec{a} + 0\vec{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$\boxed{4(d)}$ $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \middle| \begin{array}{l} x \quad\;\; - z + u = 0 \\ \quad y + z - u = 0 \end{array} \right\}$

$\boxed{(i\text{-}iii)}$ By the homogeneous subspace theorem, W is a subspace of $\mathbb{R}^4$.

Let's find a basis for W.

Let $\vec{w} = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}$ be in W.

Then,

$$\boxed{\begin{array}{l} x \quad\;\; - z + u = 0 \\ \quad y + z - u = 0 \end{array}}$$

already reduced
leading: x, y
free: z, u

So,

$$\boxed{\begin{array}{ll} x = z - u & (1) \\ y = -z + u & (2) \\ z = t & (3) \\ u = s & (4) \end{array}}$$

Thus,

$$\boxed{\begin{array}{l} u = s \\ z = t \\ y = -z + u = -t + s \\ x = z - u = t - s \end{array}}$$

So,

$$\vec{w} = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} t-s \\ -t+s \\ t \\ s \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ s \\ 0 \\ s \end{pmatrix}$$

$$= t \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, every vector $\vec{w}$ in W lies in the span of $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

These vectors are linearly independent since if

$$c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

then

$$\begin{pmatrix} c_1 - c_2 \\ -c_1 + c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$c_1 = 0, c_2 = 0.$$

Therefore, $W = \text{span}(\vec{a}, \vec{b})$ and $\beta = [\vec{a}, \vec{b}]$ is a basis for W.

And $\dim(W) = 2$ since β has 2 vectors in it.

$\boxed{(iv)}$ $W = \text{span}(\vec{a}, \vec{b})$

Thus 4 example vectors in W are:

$$1 \cdot \vec{a} + 0 \cdot \vec{b} = 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$0 \cdot \vec{a} + 1 \cdot \vec{b} = 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$2 \cdot \vec{a} + 1 \cdot \vec{b} = 2 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

$$5 \cdot \vec{a} - 5 \cdot \vec{b} = 5 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} - 5 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \\ 5 \\ -5 \end{pmatrix}$$